Growth and Decay – some examples

To review the formula:

Differential Equation of Proportional Change $\frac{dy}{dt} = k\,t$ Where: y is the amount of material present at time t k is the growth constant.
Law of Exponential Change $y = y_0 e^{k\,t}$ Where: y is the amount of material present at time t, y_0 is the amount of material originally present, k is the growth constant

It probably makes sense to you that the **k in our equation** will be some negative constant when we are dealing with the decay of a substance and a positive constant when we have the growth of a substance. Each material or substance will have its own **rate constant, k.** Always remember that the **y stands for the amount that is <u>present</u> at time t**, not the part which is gone!

Let's look at some problems. Each one illustrates a different use of the Law of Exponential Change, so it will be worth your time to look at each one. As we go through them you will notice how important it is that you are comfortable with the rules of algebra and exponents.

A: We are given an equation relating the rate at which electricity is draining from a capacitor and the passage of time. We need to a) solve for V and then b) find how long it will take for the voltage to drop to 10% of its original value.

$\frac{dV}{dt} = \left(-\frac{1}{40}\right)V$	This is the original relationship. If you compare it to the basic equation dy/dt = ky, you see that in this case k has been given to us and is equal to -1/40. It makes sense that k is negative since the voltage is dropping.
$V = V_0\, e^{(-1/40)t}$	a) First we solve for V by using the Law of Exponential Change to rewrite the proportional change equation.
$0.1V_0 = V_0\, e^{(-1/40)t}$ $0.1 = e^{(-1/40)t}$ $\ln 0.1 = -\frac{1}{40}t$ $t = -40(\ln 0.1)$ $t \approx 92.103 \text{ sec}$	b) Now we are to find the time when the voltage has dropped to 10% of its original amount. This means that we have to find t when $V = 0.1\, V_0$. Divide by V_0. Rewrite in log form. Solve for t. If you decide to change to a decimal (you don't have to), remember to give the answer correct to three places (AP rules!).

B: We have a population of bacteria which is increasing exponentially over time. At t = 3, y = 10,000. At t = 5, y = 40,000. Find the amount initially present, y_0.

$10,000 = y_0\, e^{3k}$ $40,000 = y_0\, e^{5k}$	Usually when you have to find y_0 you are faced with solving two equations simultaneously because you know neither y_0 nor k. So let's set up two equations with the information we are given for t and y.
$10,000e^{-3k} = y_0$ Therefore: $40,000 = \left(10,000e^{-3k}\right)e^{5k}$ $40,000 = 10,000e^{2k}$	I am going to solve for y_0 in the first equation and then substitute into the second equation.
$4 = e^{2k}$ $2 = e^{k}$	Now solve for e^k (I took the square root of each side of the equation.)
$10,000 = y_0\left(e^k\right)^3$ $10,000 = y_0\,(2)^3$ $y_0 = \frac{10,000}{8} = 1250$ bacteria initially present	Put e^k back into the first equation and solve for y_0. Don't forget your units!

C: In this problem we have oil being pumped from wells, which is decreasing the underground supply. It decreases at the rate of 10% per year and we want to know when it will be at 1/5 of its present (initial) value. In other words, we want to know the time t when the amount present equals 1/5 y_0 or 0.2 y_0.

$0.9\,y_0 = y_0\,e^{k*1}$ $0.9\,y_0 = y_0\,e^{k}$	We will have to find k, and to do that we will use the information given: at t = 1, y = .9 y_0 (in one year the supply will be 90% of its original amount - a decrease of 10%).
$0.9 = e^{k}$ ln 0.9 = k	Divide by y_0 and then write the equation in log form to solve for k.
$y = y_0\,e^{(\ln 0.9)t}$	Now we have an equation that represents the decrease in oil supply from the wells.
$0.2\,y_0 = y_0\,e^{(\ln 0.9)t}$ $0.2 = e^{(\ln 0.9)t}$ $\ln 0.2 = (\ln 0.9)t$ $t = \dfrac{\ln 0.2}{\ln 0.9} \approx 15.275 \text{ years}$	Find the time when y = 0.2 y_0. Divide by the y_0. Put it in log form and solve for t. Notice that as soon as I use my calculator to evaluate the quotient I have an approximate answer. You don't have to put the answer in decimal form, but if you do, give it correct to 3 places (no rounding).